1 The Bellman Equation

As explained above, Elaine's value function is an optimal choice between the flow payoff Q_t and the discounted continuation value function. Since the continuation value function is uncertain, as future quality draws are from some distribution, we must use the expectation operator in that piece of the maximization problem. Elaine's value function is then:

$$V(Q_t) = max\{Q_t, \delta EV(Q_{t+1})\}$$

This recursive relationship between value functions of different time periods. This is the Bellman Equation of lore!

2 Solving for Spongeworthiness

We are in a threshold rule problem as previously explained. This means that the value function is of the following form:

$$V(Q_t) = \begin{cases} Q_t & \text{if } Q_t > Q^* \\ Q^* & \text{if } Q_t \leq Q^* \end{cases}$$

To solve for the threshold, Elaine must be indifferent.....so we set the following equality to solve...:

$$Q^* = \delta EV(Q_{t+1})$$

$$Q^* = \delta(\int_0^{Q^*} Q^* dQ_{t+1} + \int_{Q^*}^1 Q_{t+1} dQ_{t+1})$$

$$Q^* = \delta([Q^* Q_{t+1}]\Big|_0^{Q^*} + [\frac{(Q_{t+1})^2}{2}]\Big|_{Q^*}^1)$$

$$Q^* = \delta(\frac{Q^{*2}}{2} + \frac{1}{2})$$

$$\frac{\delta Q^{*2}}{2} - Q^* + \frac{\delta}{2} = 0$$

$$Q^* = \frac{1}{\delta} \pm \sqrt{\frac{1}{\delta^2} - 1}$$

The positive root yields a $Q^* > 1$, which would mean that Elaine never uses the sponge. This cannot be the optimal policy, so we elimate this root. In effect, we end up with the following solution:

$$Q^* = \frac{1}{\delta} - \sqrt{\frac{1}{\delta^2} - 1}$$

3 Interpreting Spongeworthiness Solution

$$V(Q_t) = max\{Q_t, Q^*\} = max\{Q_t, \delta EV(Q_{t+1})\}$$

As δ approaches 1 (super patient), Q^* approaches 1, meaning Elaine will accept no one but the best possible man. At the other extreme, as δ approaches 0 (super impatient), Q^* approaches 0, meaning Elaine will immediately sleep with whoever presents himself.